J. Fluid Mech. (1989), *vol.* 203, *pp.* 289–305
Printed in Great Britain

The Dean equations extended to a helical pipe flow

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(Received 30 December 1987 and in revised form 19 December 1988)

In this paper the Dean (1928) equations are extended to the case of a helical pipe flow, and it is shown that they depend not only on the Dean number K but also on a new parameter $\lambda/\mathscr{R}$, where λ is the ratio of the torsion τ to the curvature κ of the pipe axis and $\mathscr{R}$ the Reynolds number referred in the usual way to the pipe radius a and to the equivalent maximum speed in a straight pipe under the same axial pressure gradient. The fact that the torsion has no first-order effect on the flow is confirmed, but it is shown that this is peculiar to a circular cross-section. In the case of an elliptical cross-section there is a first-order effect of the torsion on the secondary flow, and in the limit $\lambda/\mathscr{R} \rightarrow \infty$ (twisted pipes, provided only with torsion), the first-order 'displacement' effect of the walls on the secondary flow, analysed in detail by Choi (1988), is recovered.

Different systems of coordinates and different orders of approximations have recently been adopted in the study of the flow in a helical pipe. Thus comparisons between the equations and the results presented in different reports are in some cases difficult and uneasy. In this paper the extended Dean equations for a helical pipe flow recently derived by Kao (1987) are converted to a simpler form by introducing an appropriate modified stream function, and their equivalence with the present set of equations is recovered. Finally, the first-order equivalence of this set of equations with the equations obtained by Murata *et al.* (1981) is discussed.

1. Introduction

The flow in curved pipes is very complex. Perhaps the most interesting consequence of the curvature is the development of a secondary flow consisting of a pair of counter-rotating vortices as represented in figure 1. The analysis of this flow started in 1927 with the paper of Dean, in which the flow in a toroidal system of coordinates r, ξ was studied. Since then a lot of work has been done, but the effect of the torsion has scarcely been examined, and the eventual helicity of the coils was usually neglected. The experiments are usually conducted in helical coils (see, for example, Lin & Tarbell 1980), so that it is important to evaluate the effect of the torsion on the secondary flow. This problem was discussed recently by Murata *et al.* (1981), Wang (1981), Germano (1982) and Kao (1987), and they obtained, using different orthogonal and non-orthogonal systems of coordinates, solutions for the steady flow in a helical pipe with constant curvature κ and torsion τ (see figure 2).

Both Wang and Murata *et al.* studied perturbation solutions in the parameter $\epsilon = \kappa a$, where a is the radius of the pipe, but while Wang found a first-order effect of the torsion on the secondary flow comparable with the effect of the curvature, Murata *et al.* found no such effect, and this result was confirmed recently by Kao. Germano showed that the secondary flow described by Wang is not projected in a normal

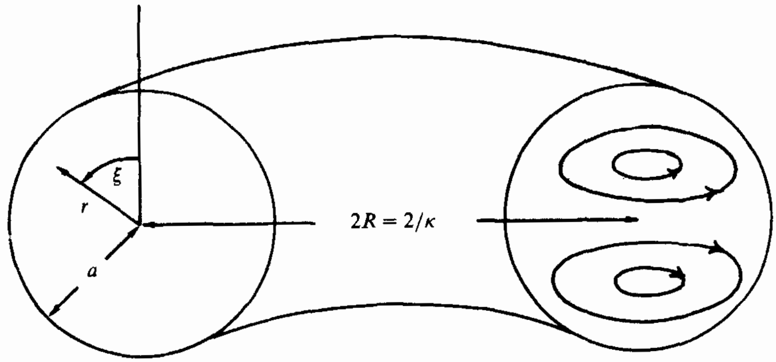

FIGURE 1. Coordinate system and secondary flow in a toroidal pipe.

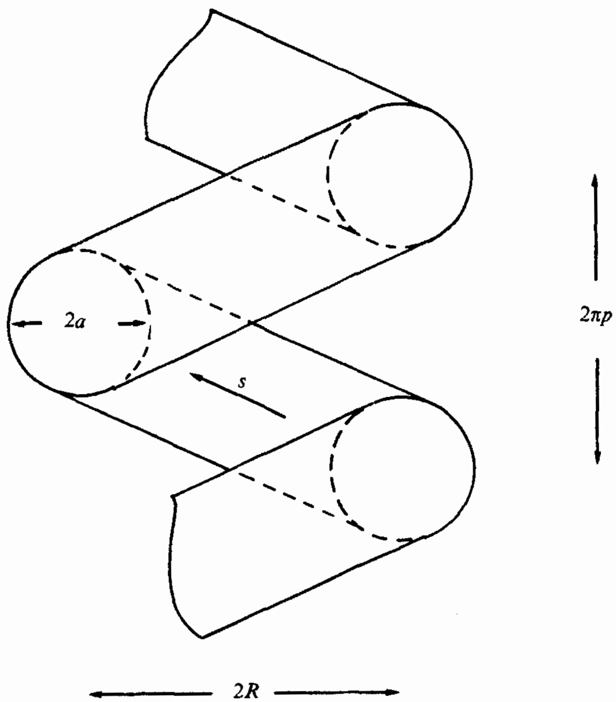

FIGURE 2. Helical pipe. Curvature $\kappa = R/(R^2+p^2)$, torsion $\tau = (p/R^2+p^2)$.

plane, and by using an orthogonal system of coordinates recovered the result of Murata *et al.*

The study of Germano (1982) was conducted using the complete Navier–Stokes equations and the expansion of the solution was in terms of the parameter ϵ. In the present paper we shall apply to these equations the standard procedure that leads to the Dean approximation (see Berger, Talbot & Yao 1983), and the Dean equations will be extended to a helical pipe. The results will be compared with those of Murata *et al.* and with those of Kao (the analysis of Wang was also conducted on the complete equations expanded in ϵ), and the dependence on the Dean number, the Reynolds number and the ratio $\lambda = \tau/\kappa$, torsion to curvature, will be discussed. The fact that the torsion has no first-order effect on helical pipes of circular cross-section

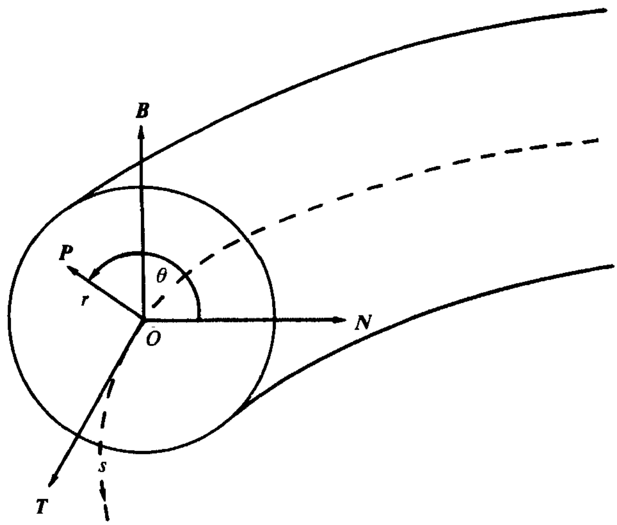

FIGURE 3. Non-orthogonal helical coordinate system.

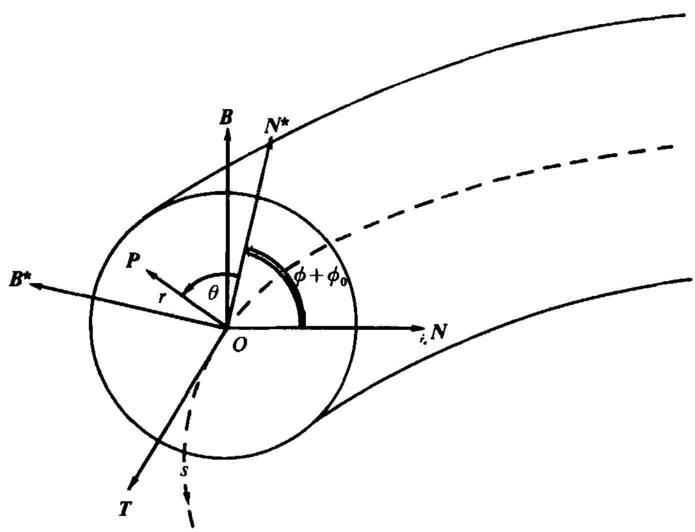

FIGURE 4. Orthogonal helical coordinate system.

will be tested once more, and its effect in the case of an elliptical cross-section will be examined. The secondary flow in a toroidal pipe of elliptical cross-section was studied by Cuming (1952), and in this paper his results are extended to a helical pipe in order to explore in this case also the eventual effect of the torsion.

2. The coordinate system and the extended Dean equations

In the study of the flow in a pipe whose axial line is a generic spatial line it is important to choose an appropriate system of coordinates. Let us assume that we know the curvature and the torsion of the axial line as functions of the arclength s, $\kappa(s)$ and $\tau(s)$. It is shown in Germano (1982) that we can introduce two different systems of coordinates, represented in figures 3 and 4. These systems of coordinates

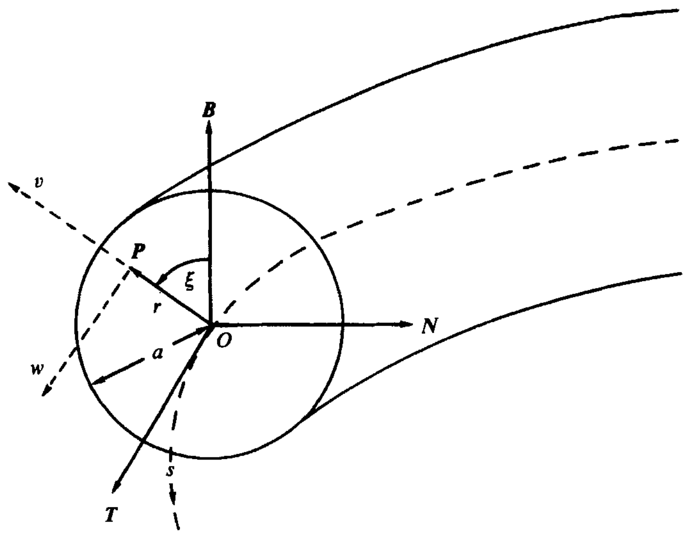

FIGURE 5. Reference system for helically symmetric solutions.

are referred to the intrinsic triad of unit vectors, and $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$ are the tangent, the normal and the binormal to the axial line. The first system of coordinates, figure 3, is non-orthogonal, and a generic point in a normal section of the pipe is identified by the arclength s and by the polar coordinates r and θ, where θ is a polar angle referred to the normal $\boldsymbol{N}$. The metric is given by the expression

$$\mathrm{d}\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{P} = [(1-\kappa r\cos\theta)^2+\tau^2r^2](\mathrm{d}s)^2+(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2+2\tau r^2\,\mathrm{d}s\,\mathrm{d}\theta.$$

In the second system of coordinates, figure 4, the polar angle θ is referred to a rotated unit vector $\boldsymbol{N}^*$, and the rotation is given by the angle $\phi+\phi_0$, where ϕ is given by the integral

$$\phi = -\int_{s_0}^{s}\tau(s')\,\mathrm{d}s',$$

and ϕ_0 is an arbitrary constant angle. In this case the metric is given by the expression

$$\mathrm{d}\boldsymbol{P}\cdot\mathrm{d}\boldsymbol{P} = [1-\kappa r\cos(\theta+\phi+\phi_0)]\,(\mathrm{d}s)^2+(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2$$

and it is easy to see that when $\tau = 0$ this system of coordinates reduces to the usual toroidal one, while it reduces to the cylindrical one when $\kappa = \tau = 0$. This system of coordinates is well known to researchers involved in studies on hydromagnetic equilibria. It was first introduced by Mercier (1963) and was extensively used by Solove'v & Shafranov (1970) in their computations of plasma confinement in closed magnetic systems. In Germano (1982) the author has written the Navier–Stokes equations for an incompressible viscous fluid in this orthogonal system of coordinates, and the constant angle ϕ_0 was assumed equal to $\frac{1}{2}\pi$. Let us use the same notation as Germano (1982):

$$\tilde{s} = \frac{s}{a},\quad \tilde{r} = \frac{r}{a},\quad (\tilde{u},\tilde{v},\tilde{w}) = \left(\frac{u}{U},\frac{v}{U},\frac{w}{U}\right),\quad \tilde{p} = \frac{p}{U^2},$$

$$\epsilon = \kappa a,\quad \lambda = \frac{\tau}{\kappa},\quad \mathscr{R} = \frac{Ua}{\nu},$$

where a is the radius of the pipe, or a mean radius in the case of a non-circular cross-section, s the length along the axis, r the distance from the axis in a normal section, p and ν the kinematic pressure and viscosity, $\mathscr{R}$ the Reynolds number, u the velocity along the tube, v and w the polar components of the velocity in a normal section (see figure 5) and U a reference velocity, usually given by the equivalent maximum speed in a straight pipe under the same pressure gradient. The continuity and the momentum equations are given by the expressions

$$\omega\frac{\partial\tilde{u}}{\partial\tilde{s}}+\frac{\partial\tilde{v}}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\theta}+\frac{\tilde{v}}{\tilde{r}}+\epsilon\omega[\tilde{v}\sin(\theta+\phi)+\tilde{w}\cos(\theta+\phi)]=0; \tag{1}$$

$$\left.\begin{aligned}
\tilde{\mathrm{D}}\tilde{u}+\epsilon\omega\tilde{u}[\tilde{v}\sin(\theta+\phi)+\tilde{w}\cos(\theta+\phi)] &= -\omega\frac{\partial\tilde{p}}{\partial\tilde{s}}+\frac{1}{\mathscr{R}}\left[\left(\frac{\partial}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\right)\left(\frac{\partial\tilde{u}}{\partial\tilde{r}}+\epsilon\omega\tilde{u}\sin(\theta+\phi)-\omega\frac{\partial\tilde{v}}{\partial\tilde{s}}\right)\right.\\
&\qquad\left.+\frac{1}{\tilde{r}}\frac{\partial}{\partial\theta}\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\theta}+\epsilon\omega\tilde{u}\cos(\theta+\phi)-\omega\frac{\partial\tilde{w}}{\partial\tilde{s}}\right)\right],\\
\tilde{\mathrm{D}}\tilde{v}-\frac{\tilde{w}^2}{\tilde{r}}-\epsilon\omega\tilde{u}^2\sin(\theta+\phi) &= -\frac{\partial\tilde{p}}{\partial\tilde{r}}-\frac{1}{\mathscr{R}}\left[\left(\frac{1}{\tilde{r}}\frac{\partial}{\partial\theta}+\epsilon\omega\cos(\theta+\phi)\right)\left(\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}-\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\theta}\right)\right.\\
&\qquad\left.-\omega\frac{\partial}{\partial\tilde{s}}\left(\omega\frac{\partial\tilde{v}}{\partial\tilde{s}}-\frac{\partial\tilde{u}}{\partial\tilde{r}}-\epsilon\omega\tilde{u}\sin(\theta+\phi)\right)\right],\\
\tilde{\mathrm{D}}\tilde{w}+\frac{\tilde{v}\tilde{w}}{\tilde{r}}-\epsilon\omega\tilde{u}^2\cos(\theta+\phi) &= -\frac{1}{\tilde{r}}\frac{\partial\tilde{p}}{\partial\theta}+\frac{1}{\mathscr{R}}\left[\left(\frac{\partial}{\partial\tilde{r}}+\epsilon\omega\sin(\theta+\phi)\right)\left(\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}-\frac{1}{\tilde{r}}\frac{\partial\tilde{v}}{\partial\theta}\right)\right.\\
&\qquad\left.-\omega\frac{\partial}{\partial\tilde{s}}\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\theta}+\epsilon\omega\tilde{u}\cos(\theta+\phi)-\omega\frac{\partial\tilde{w}}{\partial\tilde{s}}\right)\right],
\end{aligned}\right\} \tag{2}$$

where $$\omega=\frac{1}{1+\epsilon\tilde{r}\sin(\theta+\phi)},\quad \tilde{\mathrm{D}}=\omega\tilde{u}\frac{\partial}{\partial\tilde{s}}+\tilde{v}\frac{\partial}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}\frac{\partial}{\partial\theta}.$$

This set of equations is valid for a generic spatial curve, and we shall consider here the case of a helix, a curve with constant curvature κ and torsion τ. In this case it is possible to search for helically symmetric solutions of the general equations, solutions that physically correspond to a fully developed flow in a helical pipe. In order to do that we operate the following transformations from $\tilde{s}$, θ, $\tilde{r}$ to $\tilde{s}$, ξ, $\tilde{r}$:

$$\theta+\phi\Rightarrow\xi,\quad \frac{\partial}{\partial\tilde{s}}\Rightarrow\frac{\partial}{\partial\tilde{s}}-\epsilon\lambda\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial\theta}\Rightarrow\frac{\partial}{\partial\xi},$$

and if we set the resulting $\tilde{s}$-derivatives equal to zero, with the exception of the pressure derivative, we finally obtain the equations governing the required helically symmetric solutions:

$$\frac{\partial\tilde{v}}{\partial\tilde{r}}+\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\xi}+\frac{\tilde{v}}{\tilde{r}}+\epsilon\omega\left(\tilde{v}\sin\xi+\tilde{w}\cos\xi-\lambda\frac{\partial\tilde{u}}{\partial\xi}\right)=0, \tag{3}$$

$$\begin{aligned}\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\frac{\tilde{w}}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\xi}+\epsilon\omega\tilde{u}\left(\tilde{v}\sin\xi+\tilde{w}\cos\xi-\lambda\frac{\partial\tilde{u}}{\partial\xi}\right)&\\
=-\omega\frac{\partial\tilde{p}}{\partial\tilde{s}}+\epsilon\omega\lambda\frac{\partial\tilde{p}}{\partial\xi}+\frac{1}{\mathscr{R}}\left(\frac{1}{\tilde{r}}\frac{\partial\beta}{\partial\xi}-\frac{\partial\gamma}{\partial\tilde{r}}-\frac{\gamma}{\tilde{r}}\right),&\end{aligned} \tag{4a}$$

$$\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{r}}+\frac{\tilde{w}}{\tilde{r}}\frac{\partial \tilde{v}}{\partial \xi}-\frac{\tilde{w}^2}{\tilde{r}}-\epsilon\omega\tilde{u}\left(\tilde{u}\sin\xi+\lambda\frac{\partial \tilde{v}}{\partial \xi}\right)$$
$$=-\frac{\partial \tilde{p}}{\partial \tilde{r}}-\frac{1}{\mathscr{R}}\left[\frac{1}{\tilde{r}}\frac{\partial \alpha}{\partial \xi}+\epsilon\omega\left(\alpha\cos\xi+\lambda\frac{\partial \gamma}{\partial \xi}\right)\right], \quad (4b)$$

$$\tilde{v}\frac{\partial \tilde{w}}{\partial \tilde{r}}+\frac{\tilde{w}}{\tilde{r}}\frac{\partial \tilde{w}}{\partial \xi}+\frac{\tilde{v}\tilde{w}}{\tilde{r}}-\epsilon\omega\tilde{u}\left(\tilde{u}\cos\xi+\lambda\frac{\partial \tilde{w}}{\partial \xi}\right)$$
$$=-\frac{1}{\tilde{r}}\frac{\partial \tilde{p}}{\partial \xi}+\frac{1}{\mathscr{R}}\left[\frac{\partial \alpha}{\partial \tilde{r}}+\epsilon\omega\left(\alpha\sin\xi+\lambda\frac{\partial \beta}{\partial \xi}\right)\right], \quad (4c)$$

where

$$\omega=\frac{1}{1+\epsilon\tilde{r}\sin\xi}, \quad \alpha=\frac{\partial \tilde{w}}{\partial \tilde{r}}+\frac{\tilde{w}}{\tilde{r}}-\frac{1}{\tilde{r}}\frac{\partial \tilde{v}}{\partial \xi},$$

$$\beta=\frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \xi}+\epsilon\omega\left(\tilde{u}\cos\xi+\lambda\frac{\partial \tilde{w}}{\partial \xi}\right), \quad \gamma=-\frac{\partial \tilde{u}}{\partial \tilde{r}}-\epsilon\omega\left(\tilde{u}\sin\xi+\lambda\frac{\partial \tilde{v}}{\partial \xi}\right).$$

Let us now drop higher-order terms in ϵ from (3) and (4). This physically corresponds to considering loosely coiled pipes, and following the standard procedure we preserve the centrifugal terms and we write explicitly

$$\bar{v}=\mathscr{R}\tilde{v}, \quad \bar{w}=\mathscr{R}\tilde{w}, \quad \tilde{p}=-\frac{G}{\mathscr{R}}\tilde{s}+\epsilon\tilde{p}_1(\tilde{r},\xi),$$

where G is a constant related to the dimensional pressure gradient along the pipe by the expression

$$G=\frac{\mathscr{R}a}{\rho U^2}H,$$

where H is the pressure gradient. We now introduce the Dean number $K=2\epsilon\mathscr{R}^2$, and we obtain

$$\frac{\partial(\tilde{r}\bar{v})}{\partial \tilde{r}}+\frac{\partial}{\partial \xi}\left(\bar{w}-\frac{\lambda}{\mathscr{R}}\tfrac{1}{2}K\tilde{r}\tilde{u}\right)=0, \quad (5)$$

$$\bar{v}\frac{\partial \tilde{u}}{\partial \tilde{r}}+\frac{\partial \tilde{u}}{\partial \xi}\frac{\bar{w}-(\lambda/\mathscr{R})\tfrac{1}{2}K\tilde{r}\tilde{u}}{\tilde{r}}=G+\nabla^2\tilde{u}, \quad (6a)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \tilde{r}}+\frac{\bar{w}}{\tilde{r}}\frac{\partial \bar{v}}{\partial \xi}-\frac{\bar{w}^2}{\tilde{r}}-\tfrac{1}{2}K\tilde{u}^2\sin\xi-\frac{\lambda}{\mathscr{R}}\tfrac{1}{2}K\tilde{u}\frac{\partial \bar{v}}{\partial \xi}$$
$$=-\tfrac{1}{2}K\frac{\partial \tilde{p}_1}{\partial \tilde{r}}-\frac{1}{\tilde{r}}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{\tilde{r}}\left(\frac{\partial(\bar{w}\tilde{r})}{\partial \tilde{r}}-\frac{\partial \bar{v}}{\partial \xi}\right)\right)\right]+\tfrac{1}{2}K\frac{\lambda}{\mathscr{R}}\frac{\partial^2 \tilde{u}}{\partial \xi\,\partial \tilde{r}}, \quad (6b)$$

$$\bar{v}\frac{\partial \bar{w}}{\partial \tilde{r}}+\frac{\bar{w}}{\tilde{r}}\frac{\partial \bar{w}}{\partial \xi}+\frac{\bar{v}\bar{w}}{\tilde{r}}-\tfrac{1}{2}K\tilde{u}^2\cos\xi-\frac{\lambda}{\mathscr{R}}\tfrac{1}{2}K\tilde{u}\frac{\partial \bar{w}}{\partial \xi}$$
$$=-\tfrac{1}{2}K\frac{\partial \tilde{p}_1}{\partial \xi}+\frac{\partial}{\partial \tilde{r}}\left(\frac{1}{\tilde{r}}\left(\frac{\partial(\bar{w}\tilde{r})}{\partial \tilde{r}}-\frac{\partial \bar{v}}{\partial \xi}\right)\right)+\tfrac{1}{2}K\frac{\lambda}{\mathscr{R}}\frac{1}{\tilde{r}}\frac{\partial^2 \tilde{u}}{\partial \xi^2}, \quad (6c)$$

where higher-order terms in ϵ have been discarded and where

$$\nabla^2=\frac{\partial^2}{\partial \tilde{r}^2}+\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}+\frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial \xi^2}.$$

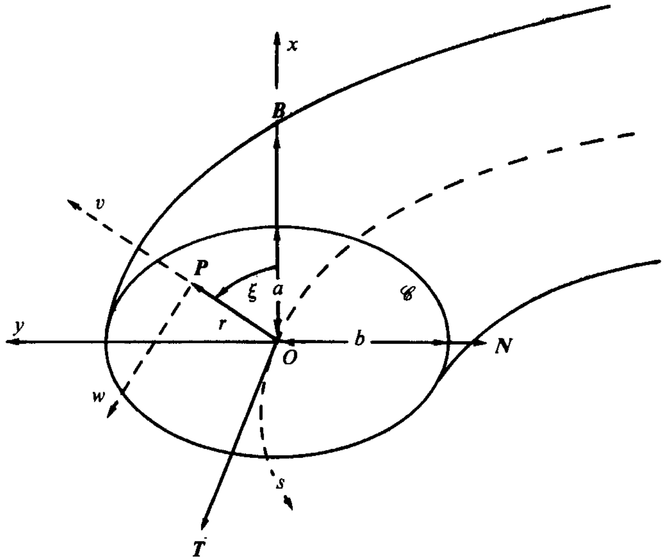

FIGURE 6. Reference system for helically symmetric solutions: elliptical cross-section with axes oriented with $\boldsymbol{N}$ and $\boldsymbol{B}$.

Equation (5) is automatically satisfied by the pseudo-stream function ψ:

$$\tilde{r}\tilde{v} = -\frac{\partial\psi}{\partial\xi}, \quad \tilde{w}-\frac{\lambda}{\mathscr{R}}\tfrac{1}{2}K\tilde{r}\tilde{u} = \frac{\partial\psi}{\partial\tilde{r}}, \tag{7}$$

and from (6) we finally obtain

$$\nabla_2\tilde{u}+G = \frac{1}{\tilde{r}}\left(\frac{\partial\psi}{\partial\tilde{r}}\frac{\partial\tilde{u}}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial\tilde{u}}{\partial\tilde{r}}\right), \tag{8}$$

$$\nabla^4\psi = \frac{1}{\tilde{r}}\left(\frac{\partial\psi}{\partial\tilde{r}}\frac{\partial}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial}{\partial\tilde{r}}\right)\nabla^2\psi+K\tilde{u}\left(\frac{\sin\xi}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\xi}-\cos\xi\frac{\partial\tilde{u}}{\partial\tilde{r}}\right)+2GK\frac{\lambda}{\mathscr{R}}+\tfrac{1}{4}K^2\left(\frac{\lambda}{\mathscr{R}}\right)^2\frac{\partial(\tilde{u}^2)}{\partial\xi}. \tag{9}$$

Equations (8) and (9) represent the extended Dean equations for a helical pipe flow. They depend not only on the Dean number K but also on the parameter $\lambda/\mathscr{R}$, and we note that the effect of the torsion is reduced to two terms, one of which is a constant, affecting the equation for the stream function ψ. If the value of this parameter is low, typically for high Reynolds numbers or for low values of the torsion, the equations reduce to the usual Dean equations. It is apparent that the effect of the torsion increases with decreasing Reynolds number.

3. First-order effect of the torsion on pipes of elliptical cross-section

Let us now consider a pipe with an elliptical cross-section, whose axes of lengths $2b$ and $2a$ are always oriented in the directions of the normal $\boldsymbol{N}$ and the binormal $\boldsymbol{B}$ to the helical axis, as in figure 6.

If we assume a as a reference length, the boundary $\mathscr{C}$ is given by the relation

$$1-\tilde{r}^2+(1-\Lambda^2)\,\tilde{r}^2\sin^2\xi = 0,$$

where $\Lambda = a/b$, and the boundary conditions on $\mathscr{C}$ for $\tilde{u}$ and ψ are

$$\tilde{u} = \psi = \frac{\partial\psi}{\partial\tilde{r}} = 0.$$

If we now introduce in (8) and (9) the usual expansions in terms of the integer powers of the Dean number K:

$$\left.\begin{aligned} \tilde{u} &= \tilde{u}_0 + K\tilde{u}_1 + K^2 u_2 + \dots, \\ \psi &= \psi_0 + K\psi_1 + K^2\psi_2 + \dots, \end{aligned}\right\} \tag{10}$$

we obtain to the first order the following sets of equations:

$$\left.\begin{aligned} \nabla^2 \tilde{u}_0 + G &= 0, \\ \psi_0 &= 0, \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} \nabla^2 \tilde{u}_1 &= \frac{1}{\tilde{r}}\left(\frac{\partial\psi_1}{\partial\tilde{r}}\frac{\partial\tilde{u}_0}{\partial\xi} - \frac{\partial\psi_1}{\partial\xi}\frac{\partial\tilde{u}_0}{\partial\tilde{r}}\right), \\ \nabla^4\psi_1 &= \tilde{u}_0\left(\frac{\sin\xi}{\tilde{r}}\frac{\partial\tilde{u}_0}{\partial\xi} - \cos\xi\frac{\partial\tilde{u}_0}{\partial\tilde{r}}\right) + 2G\frac{\lambda}{\mathscr{R}}; \end{aligned}\right\} \tag{12}$$

and if we assume as reference velocity U the maximum speed in a straight pipe under the same pressure gradient, we have

$$G = 4 - 2(1 - \varLambda^2) \tag{13}$$

and

$$\tilde{u}_0 = 1 - \tilde{r}^2 + (1 - \varLambda^2)\,\tilde{r}^2 \sin^2\xi. \tag{14}$$

The value of ψ_1 in the toroidal case, $\lambda/\mathscr{R} = 0$, was obtained by Cuming (1952), who extended the Dean results to pipes of elliptical cross-section. We indicate this value as ψ_1^* and its expression is given by

$$\psi_1^* = \tilde{u}_0^2(C_1 + C_2\tilde{r}^2\sin^2\xi + C_3\tilde{r}^2\cos^2\xi)\,\tilde{r}\cos\xi, \tag{15}$$

where C_1, C_2 and C_3 are constants depending on $\varLambda$:

$$C_1 = \frac{375 + 820\varLambda^2 + 1114\varLambda^4 + 212\varLambda^6 + 39\varLambda^8}{180(5 + 2\varLambda^2 + \varLambda^4)\,F(\varLambda)},$$

$$C_2 = -\frac{(75 + 2\varLambda^2 + 3\varLambda^4)}{180F(\varLambda)}, \quad C_3 = -\frac{(15 + 26\varLambda^2 + 39\varLambda^4)}{180F(\varLambda)},$$

$$F(\varLambda) = 35 + 84\varLambda^2 + 114\varLambda^4 + 20\varLambda^6 + 3\varLambda^8.$$

Obviously when $\varLambda = 1$ this expression for ψ_1^* reduces to the first-order Dean solution, and it is easy to see that in the case $\lambda/\mathscr{R} \neq 0$ we satisfy both the boundary conditions and the perturbed equation when

$$\psi_1 = \psi_1^* + h\tilde{u}_0^2\frac{\lambda}{\mathscr{R}}, \tag{16}$$

where h is given by

$$h = \frac{1 + \varLambda^2}{6(1 + \varLambda^4) + 4\varLambda^2}.$$

The secondary flow in the normal section is given by

$$\bar{v} = K\bar{v}_1 + K^2\bar{v}_2 + \dots, \quad \bar{w} = K\bar{w}_1 + K^2\bar{w}_2 + \dots, \tag{17}$$

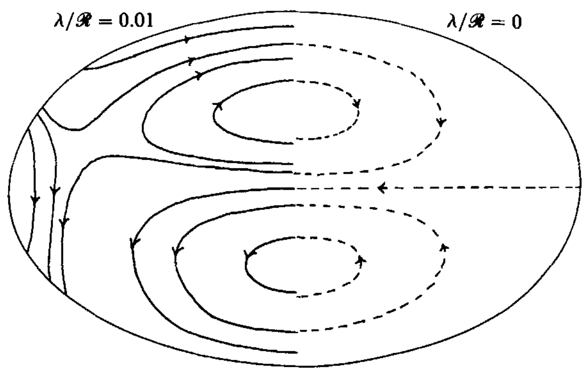

FIGURE 7. Secondary flow, $\Lambda = 0.65$, $\lambda/\mathscr{R} = 0.01$.

where
$$\left.\begin{aligned}\bar{v}_1 &= -\frac{1}{\tilde{r}}\left(\frac{\partial \psi_1^*}{\partial \xi} + 2h\tilde{u}_0 \frac{\lambda}{\mathscr{R}}\frac{\partial \tilde{u}_0}{\partial \xi}\right),\\ \bar{w}_1 &= \frac{\partial \psi_1^*}{\partial \tilde{r}} + \tilde{u}_0 \frac{\lambda}{\mathscr{R}}\left(\tfrac{1}{2}\tilde{r} + 2h\frac{\partial \tilde{u}_0}{\partial \tilde{r}}\right),\end{aligned}\right\} \tag{18}$$

and we can see that when $\Lambda = 1$, circular cross-section, we obtain
$$\bar{v}_1 = -\frac{1}{\tilde{r}}\frac{\partial \psi_1^*}{\partial \xi}, \quad \bar{w}_1 = \frac{\partial \psi_1^*}{\partial \tilde{r}},$$

so that the fact that the torsion has no first-order effect on the secondary flow (Germano 1982) is confirmed. On the other hand it is evident that when the cross-section is elliptical there is a first-order effect on the secondary flow. It is interesting to note that the perturbation to the axial velocity $\tilde{u}_1$, and by consequence the flux, do not depend in any case on the torsion at the first order. In effect, since
$$\psi_1 = \psi_1^* + h\frac{\lambda}{\mathscr{R}}\tilde{u}_0^2,$$
we obtain from (12*a*)
$$\nabla^2 \tilde{u}_1 = \frac{1}{\tilde{r}}\left(\frac{\partial \psi_1^*}{\partial \tilde{r}}\frac{\partial \tilde{u}_0}{\partial \xi} - \frac{\partial \psi_1^*}{\partial \xi}\frac{\partial \tilde{u}_0}{\partial \tilde{r}}\right),$$
which shows the independence of $\tilde{u}_1$ from the torsion. The Cartesian components of the secondary motion are explicitly given by
$$\left.\begin{aligned}v_x &= 2\epsilon U\mathscr{R}(1-x^2-\Lambda^2y^2)\Big[-2C_2x(1-x^2-\Lambda^2y^2)\\ &\qquad + 4\Lambda^2\left(h\frac{\lambda}{\mathscr{R}} + C_1x + C_2xy^2 + C_3x^3\right) - \frac{\lambda}{2\mathscr{R}}\Big]y,\\ v_y &= 2\epsilon U\mathscr{R}(1-x^2-\Lambda^2y^2)\Big[(C_1 + C_2y^2 + 3C_3x^2)(1-x^2-\Lambda^2y^2)\\ &\qquad - 4x\left(h\frac{\lambda}{\mathscr{R}} + C_1x + C_2xy^2 + C_3x^3\right) + \frac{\lambda}{2\mathscr{R}}x\Big],\end{aligned}\right\} \tag{19}$$

where x and y are dimensionless with respect to a. In the limit $\lambda/\mathscr{R} = 0$ we recover the first-order result in ϵ of Cuming and in figure 7 the pattern of the secondary flow

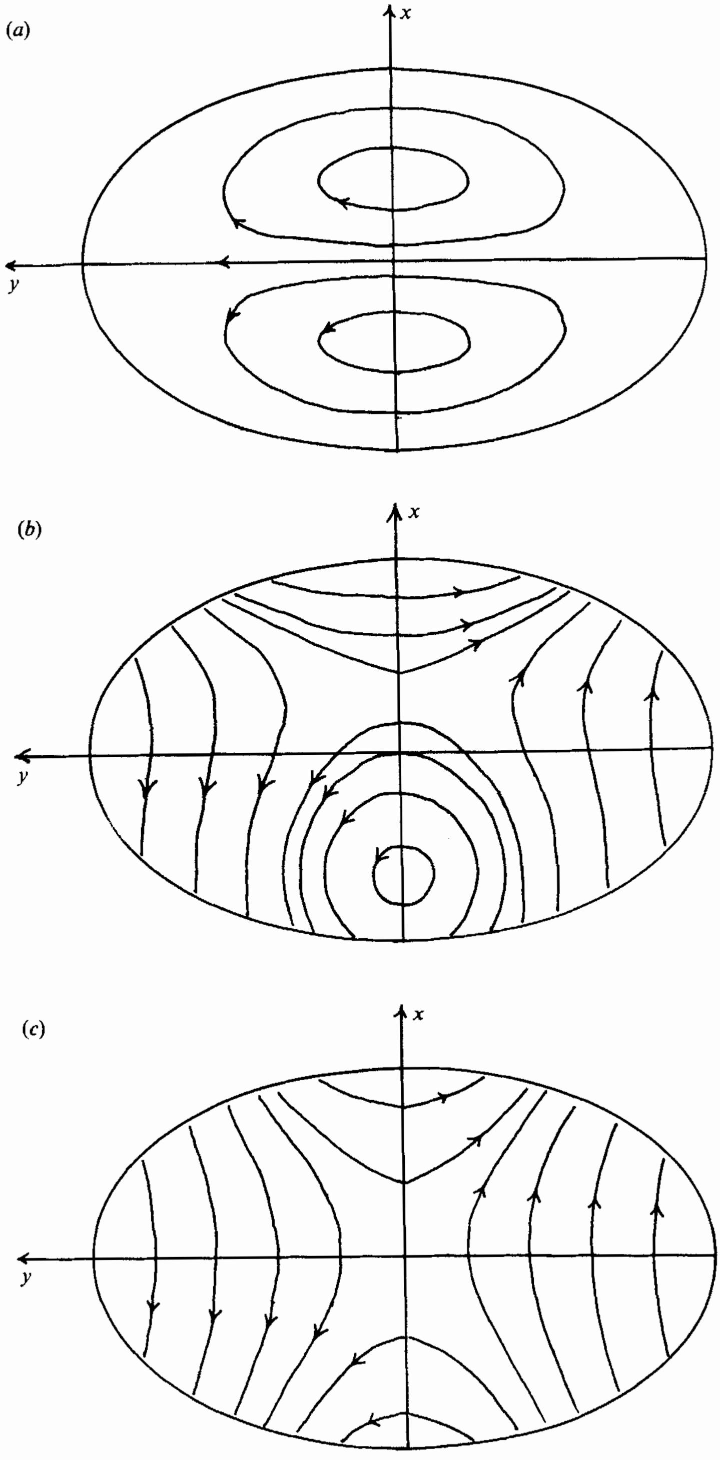

FIGURE 8. Secondary flow, $\Lambda = 0.6$: (a) $\lambda/\mathscr{R} = 0$; (b) 0.1; (c) 1.

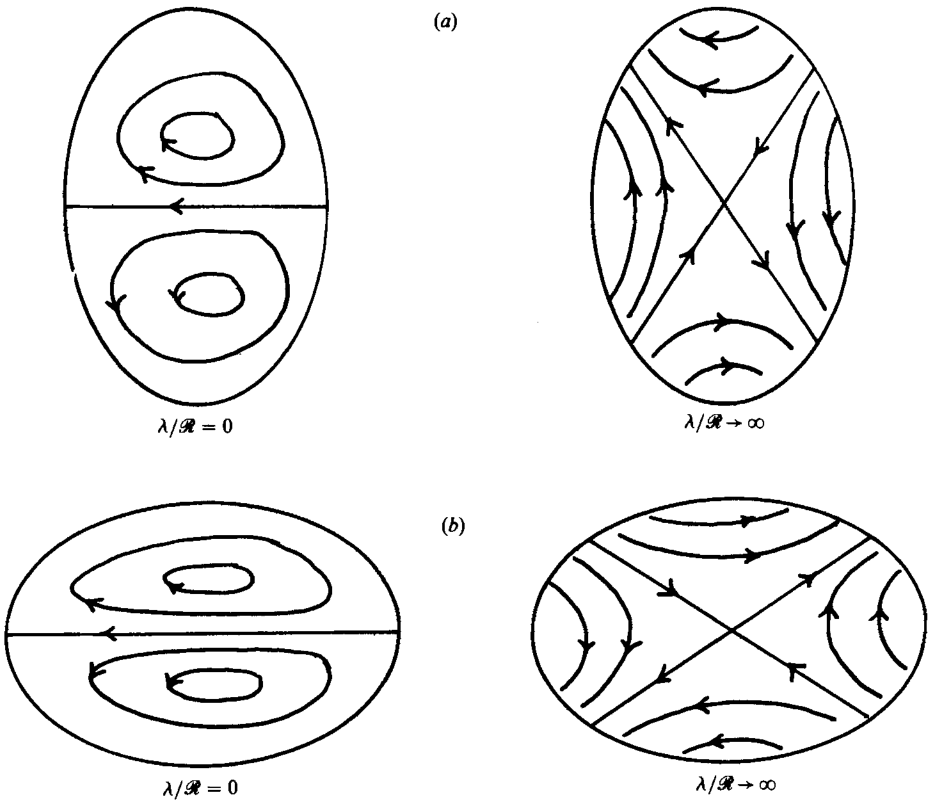

FIGURE 9. Secondary flow, $\lambda/\mathscr{R} = 0$, $\lambda/\mathscr{R} \to \infty$: (*a*) $\varLambda > 1$; (*b*) $\varLambda < 1$.

for a moderate value of $\lambda/\mathscr{R}$ and for an aspect ratio $\varLambda = 0.65$ is presented. We note a small but significant modification of the two-cell structure. This becomes much more marked as $\lambda/\mathscr{R}$ increases, as seen in figure 8 where a sequence for $\lambda/\mathscr{R} = 0, 0.1, 1$ (and for $\varLambda = 0.6$) is produced. The case $\lambda/\mathscr{R} \to \infty$ is particularly interesting to discuss and it is easy to see from (19) that for $\varLambda = 0.6$, owing to their sensitivity to the aspect ratio $\varLambda$, this limit is practically reached when $\lambda/\mathscr{R} = 1$. It physically corresponds to the case of pipes provided only with torsion, twisted pipes, and in this limit the patterns of the induced secondary flow can be calculated very easily from (19). In this limit the Cartesian components of the secondary motion are given by

$$\left.\begin{aligned} v_x &= U\tau a(1-x^2-\varLambda^2y^2)\frac{\varLambda^4+2\varLambda^2-3}{3\varLambda^4+2\varLambda^2+3}y, \\ v_y &= U\tau a(1-x^2-\varLambda^2y^2)\frac{3\varLambda^4-2\varLambda^2-1}{3\varLambda^4+2\varLambda^2+3}x, \end{aligned}\right\} \tag{20}$$

and we can derive that

$$\frac{v_x}{v_y} = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{3+\varLambda^2}{1+3\varLambda^2}\frac{y}{x}.$$

It is easy to see that the apparent streamlines of the secondary flow in a twisted pipe are, to first order, hyperbolae

$$y^2 - qx^2 = \text{constant},$$

where
$$q = \frac{1+3\Lambda^2}{3+\Lambda^2}$$

and the essential features of these limiting patterns are given in figure 9. We see that they are very different from the Dean–Cuming circulatory flow. The pure effect of the torsion is as if a saddle flow were induced in the normal planes, whose direction depends on Λ. The flow is reversed when $\Lambda = 1$, and the boundaries apparently act as sinks and sources for the secondary flow. These results are in agreement with similar results obtained by Todd (1977), Kotorynski (1986) and by Choi (1988) and we refer also to the recent paper of Todd (1986) for a general analysis of the steady laminar flow through thin curved pipes. In Choi's paper the steady fully developed flow through a slowly twisted pipe of elliptical cross-section is analysed by perturbation methods, starting from the complete Navier–Stokes equations. The expansion is conducted up to the second order in ϵ, and the first-order terms of the secondary velocities agree perfectly with our results in the case $\lambda/\mathscr{R} \rightarrow \infty$ expressed by the relations (20). The unexpected form of the secondary flow, so different from the two-cell structure, and the role of the walls that apparently act as sources and sinks, are clearly discussed by Choi in terms of a first-order 'displacement' effect of the non-circular cross-section in the case of a pipe provided with torsion. If we refer to figure 8(*c*) we see that the torsional rotation of the walls actively pushes the flow in the direction of torsion, and its effect is similar to that of an actual wall displacement. Obviously the real trajectories do not go from one wall to another, but from $-\infty$ to ∞, and in Choi's paper they are calculated by integrating with respect to time the components of velocity relative to the rotating elliptic cross-section. We note finally that the limiting equations obtained by (7), (8) and (9) when $\lambda/\mathscr{R} \rightarrow \infty$ are given by

$$\tilde{r}\bar{v} = -\frac{\partial \psi}{\partial \xi}, \quad \bar{w} = \frac{\partial \psi}{\partial \tilde{r}} + T\tilde{r}\tilde{u}, \tag{21}$$

$$\nabla^2 \tilde{u} + G = \frac{1}{\tilde{r}}\left(\frac{\partial \psi}{\partial \tilde{r}}\frac{\partial \tilde{u}}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \tilde{u}}{\partial \tilde{r}}\right), \tag{22}$$

$$\nabla^4 \psi = \frac{1}{\tilde{r}}\left(\frac{\partial \psi}{\partial \tilde{r}}\frac{\partial}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial}{\partial \tilde{r}}\right)\nabla^2 \psi + 4GT + T^2\frac{\partial(\tilde{u}^2)}{\partial \xi}, \tag{23}$$

where
$$T = \frac{\lambda}{\mathscr{R}}\tfrac{1}{2}K = \tau a \mathscr{R}.$$

We see that in this case the equations depend only on the parameter T, and in the case of a circular cross-section they are identically satisfied by the unperturbed Poiseuille flow. Obviously the 'pure' effect of torsion on a twisted tube of circular cross-section is zero at all orders.

4. Comparisons among different authors

The fully developed laminar flow in a helical pipe has been studied recently in four papers. In two of these, Wang (1981) and Murata *et al.* (1981), the flow equations are deduced in a system of non-orthogonal coordinates, while in the other two, Germano (1982) and Kao (1987), an orthogonal system of coordinates has been adopted. Some controversy exists concerning the results and in this section we try to compare the different equations obtained in these papers.

Let us consider first the article by Kao. Like the present author he adopts an orthogonal system of coordinates, and in the limit $\epsilon \to 0$ his equations are

$$\frac{\partial v}{\partial \alpha}+\frac{\partial}{\partial r}(ru) = -\beta^{\frac{1}{2}}K^{\frac{1}{2}}r\frac{\partial \bar{w}}{\partial \alpha}, \tag{24}$$

$$\nabla^2\bar{w}-\left(u\frac{\partial \bar{w}}{\partial r}+\frac{v}{r}\frac{\partial \bar{w}}{\partial \alpha}\right) = -4+\beta^{\frac{1}{2}}K^{\frac{1}{2}}\bar{w}\frac{\partial \bar{w}}{\partial \alpha}, \tag{25}$$

$$\begin{aligned}\nabla^2\Omega-\left(u\frac{\partial \Omega}{\partial r}+\frac{v}{r}\frac{\partial \Omega}{\partial \alpha}\right) &= K\bar{w}\left(\sin\alpha\frac{\partial \bar{w}}{\partial r}+\frac{\cos\alpha}{r}\frac{\partial \bar{w}}{\partial \alpha}\right)\\ &\quad +\beta^{\frac{1}{2}}K^{\frac{1}{2}}\left(\bar{w}\frac{\partial \Omega}{\partial \alpha}-\Omega\frac{\partial \bar{w}}{\partial \alpha}\right)-\beta^{\frac{1}{2}}k^{\frac{1}{2}}\left(\frac{1}{r}\frac{\partial \bar{w}}{\partial \alpha}\frac{\partial u}{\partial \alpha}-\frac{\partial \bar{w}}{\partial r}\frac{\partial v}{\partial \alpha}\right).\end{aligned} \tag{26}$$

In these equations K is the Dean number, Ω is the dimensionless vorticity for the secondary flow and $\beta^{\frac{1}{2}}$ is given by

$$\beta^{\frac{1}{2}} = \frac{a\tau}{(2\epsilon)^{\frac{1}{2}}}. \tag{27}$$

The radial and azimuthal components of the velocity u and v are made dimensionless with respect to ν/a, the axial component $\bar{w}$ with respect to

$$K^{\frac{1}{2}}(2\epsilon)^{-\frac{1}{2}}\frac{\nu}{a}.$$

r and α are polar coordinates in a normal plane, and the radial coordinate r is referred to the pipe radius a.

At this point Kao introduces the following stream function ψ (his equation (15)):

$$\left.\begin{aligned} u &= \frac{1}{r}\frac{\partial \psi}{\partial \alpha}-\beta^{\frac{1}{2}}K^{\frac{1}{2}}\frac{1}{r}\int_0^r r\frac{\partial \bar{w}}{\partial \alpha}\mathrm{d}r,\\ v &= -\frac{\partial \psi}{\partial r},\end{aligned}\right\} \tag{28}$$

which identically satisfies (24) and which, when introduced in (25) and (26), finally gives (his equations (16) and (17))

$$\nabla^2\bar{w}+4 = \frac{1}{r}\left(\frac{\partial \psi}{\partial \alpha}\frac{\partial \bar{w}}{\partial r}-\frac{\partial \psi}{\partial r}\frac{\partial \bar{w}}{\partial \alpha}\right)+\beta^{\frac{1}{2}}K^{\frac{1}{2}}\left(\bar{w}\frac{\partial \bar{w}}{\partial \alpha}-\frac{1}{r}\frac{\partial \bar{w}}{\partial r}\int_0^r r\frac{\partial \bar{w}}{\partial \alpha}\mathrm{d}r\right), \tag{29}$$

$$\begin{aligned}\nabla^4\psi &= \frac{1}{r}\left(\frac{\partial \psi}{\partial \alpha}\frac{\partial}{\partial r}-\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \alpha}\right)\nabla^2\psi+\beta^{\frac{1}{2}}K^{\frac{1}{2}}\left[\frac{1}{r^4}\int_0^r r\left(4\frac{\partial^2 \bar{w}}{\partial \alpha^2}+\frac{\partial^4 \bar{w}}{\partial \alpha^4}\right)\mathrm{d}r\right.\\ &\quad \left.+\frac{2}{r^4}\frac{\partial \psi}{\partial \alpha}\int_0^r r\frac{\partial^2 \bar{w}}{\partial \alpha^2}\mathrm{d}r+\frac{1}{r^3}\frac{\partial \psi}{\partial r}\int_0^r r\frac{\partial^3 \bar{w}}{\partial \alpha^3}\mathrm{d}r-\frac{1}{r}\frac{\partial}{\partial r}\nabla^2\psi\int_0^r r\frac{\partial \bar{w}}{\partial \alpha}\mathrm{d}r\right]\\ &\quad +\beta^{\frac{1}{2}}K^{\frac{1}{2}}\left[\frac{1}{r}\frac{\partial^3 \bar{w}}{\partial r\,\partial \alpha^2}-\frac{1}{r^2}\left(2+\frac{\partial \psi}{\partial \alpha}\right)\frac{\partial^2 \bar{w}}{\partial \alpha^2}-\left(\frac{\partial \bar{w}}{\partial \alpha}-\bar{w}\frac{\partial}{\partial \alpha}\right)\nabla^2\psi+\frac{1}{r^2}\frac{\partial \bar{w}}{\partial \alpha}\frac{\partial^2 \psi}{\partial \alpha^2}+\frac{\partial \bar{w}}{\partial r}\frac{\partial^2 \psi}{\partial r\,\partial \alpha}\right]\\ &\quad +K\left[-\bar{w}\left(\sin\alpha\frac{\partial \bar{w}}{\partial r}+\frac{\cos\alpha}{r}\frac{\partial \bar{w}}{\partial \alpha}\right)+\frac{\beta}{r^2}\int_0^r r\frac{\partial \bar{w}}{\partial \alpha}\mathrm{d}r\left(\frac{\partial^2 \bar{w}}{\partial \alpha^2}-\frac{2}{r^2}\int_0^r r\frac{\partial^2 \bar{w}}{\partial \alpha^2}\mathrm{d}r\right)-\frac{\beta}{r^2}\bar{w}\int_0^r r\frac{\partial^3 \bar{w}}{\partial \alpha^3}\mathrm{d}r\right].\end{aligned} \tag{30}$$

The pseudo-stream function ψ introduced by Kao is well suited to the study of the complete equations, but in the case of this reduced set of equations a different choice introduces considerable simplifications. Let us introduce a modified stream function φ given by

$$ru = -\frac{\partial\varphi}{\partial\alpha}, \quad v - Ar\bar{w} = \frac{\partial\varphi}{\partial r}, \tag{31}$$

where $A = -\beta^{\frac{1}{2}}K^{\frac{1}{2}}$. The continuity equation (24) is automatically satisfied, and it is easy to see (Appendix A) that (29) and (30) reduce exactly to the extended Dean equations (8) and (9) derived in the present paper:

$$\nabla^2\bar{w} + 4 = \frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\frac{\partial\bar{w}}{\partial\alpha} - \frac{\partial\varphi}{\partial\alpha}\frac{\partial\bar{w}}{\partial r}\right), \tag{32}$$

$$\nabla^4\varphi = \frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\frac{\partial}{\partial\alpha} - \frac{\partial\varphi}{\partial\alpha}\frac{\partial}{\partial r}\right)\nabla^2\varphi + K\bar{w}\left(\sin\alpha\frac{\partial\bar{w}}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial\bar{w}}{\partial\alpha}\right) + 16A + A^2\frac{\partial(\bar{w}^2)}{\partial\alpha}, \tag{33}$$

apart from the different symbols and the fact that $\alpha = \xi - \frac{1}{2}\pi$.

Let us now consider the paper by Murata *et al.* The equations obtained by these authors are given by the expressions (18)–(20) of their article, see Appendix B, and they are exactly the equations derived in this paper, apart from the term $A^2\,\partial(\bar{w}^2)/\partial\alpha$ that appears in the present equation (33), which in their order of approximation is lost. Their equations were derived in a non-orthogonal system of coordinates but they considered the opportunity of converting their results to an orthogonal form: 'The coordinates axes used here...are oblique except for the pipe center line. Accordingly u, v and w are expressed as the velocity components along these oblique axes. It is more convenient for understanding to express the velocity components along the axes r, φ (α for Kao), and the axis perpendicular to r–φ plane everywhere in a pipe cross section' (Murata *et al.*, p. 357). As a consequence they operate the transformation from their oblique components to the orthogonal components, and their equations and the present extended Dean equations are the same apart from a term discarded in their order of approximation. As regards, finally, the results of Kao we notice that, apart from a sign that depends on the fact that his positive torsion is a negative torsion for the present author, the quantity $A = -\beta^{\frac{1}{2}}K^{\frac{1}{2}}$ corresponds to the quantity $(\lambda/\mathcal{R})\frac{1}{2}K$ of our equations. As a consequence Kao expands his solutions in terms of the semi-integer powers of the Dean number K, and the parameter $\beta^{\frac{1}{2}}$ appears, while our expansions are in terms of the integer powers of the Dean number K, and the discussion of the torsion effect is conducted in terms of the parameter $\lambda/\mathcal{R}$.

5. Conclusions

The Dean equations extended to a helical pipe flow depend not only on the Dean number K but also on the parameter $\lambda/\mathcal{R}$, where λ is the ratio of torsion to curvature and $\mathcal{R}$ the Reynolds number. The importance of the terms induced by the torsion of the pipe decreases by increasing $\mathcal{R}$, so that one important conclusion is that the effect of the torsion is more evident at low Reynolds numbers.

In this paper the extended Dean equations are derived from the complete Navier–Stokes equations and are studied in terms of expansions in the integer powers of the Dean number K. When $\lambda/\mathcal{R} = 0$ we recover the classical solutions recently studied in detail by Van Dyke (1978), and in the limit of the first-order terms no first-order effect of the torsion has been shown. All that confirms a previous result (Germano 1982) and in this paper it is shown that this result is peculiar to the circular cross-section. In the case of a helical pipe of elliptical cross-section there is a first-order effect of the torsion on the secondary flow, and in the limit of $\lambda/\mathscr{R} \to \infty$, very low Reynolds numbers, it consists of a saddle flow in normal planes. These results are confirmed by similar results obtained by different authors following different procedures. A more complex expression of the extended Dean equations derived by Kao (1987) has been shown in this paper as equivalent to the present formulation, while the relations of Murata *et al.* (1981) differ from the present equations by a term in $(\lambda/\mathscr{R})^2$ retained both by Kao and the present author.

The comparison among the different papers is slightly uneasy because of the different symbols, coordinates, parameters and kind of expansions used by the different authors. Kao in particular expands the perturbative solutions in terms of the semi-integer powers of K, and the parameter $\beta^{\frac{1}{2}}$ that appears in its equations corresponds in terms of the present symbols to the quantity $(\lambda/\mathscr{R})\frac{1}{2}K^{\frac{1}{2}}$. The present form of the extended Dean equations and the proposed parameters are, in the opinion of the author, simply connected to the physics and the geometry of the problem. The expansion in terms of the integer powers of the Dean number permits a direct comparison of the results with the toroidal case, and the extension of the analysis to higher-order terms by computer should be conducted in a similar way to the usual extensions, see Van Dyke (1978). In the opinion of the author it is advisable in future studies to use the modified stream function (7) introduced in this paper and the simple extended Dean equations (8) and (9) that follow. In this confusing scenario of orthogonal and non-orthogonal coordinates, higher and lower order of approximations, this form of the extended Dean equations represents, in the opinion of the author, a reference point reached and confirmed by different and independent researches and methods. We stress finally that a peculiar advantage of the orthogonal system of coordinates is due to the fact that they can be easily generalized to a generic spatial axis and to a generic system of coordinates in normal planes (Germano & Oggiano 1985).

The author acknowledges financial support from the Italian Ministry of Education.

Appendix A

Let us express the vorticity Ω in terms of the new modified stream function φ. We have

$$\Omega = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \alpha} = \nabla^2\varphi + 2A\bar{w} + Ar\frac{\partial \bar{w}}{\partial r}, \tag{A 1}$$

where
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2},$$

and if we now introduce this expression in (26), (equation (14) of Kao 1987), we obtain

$$\begin{aligned}\nabla^4\varphi + 2A\nabla^2\bar{w} + A\nabla^2\left(r\frac{\partial \bar{w}}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial \varphi}{\partial \alpha}\frac{\partial}{\partial r} - \frac{\partial \varphi}{\partial r}\frac{\partial}{\partial \alpha}\right)\left(\nabla^2\varphi + 2A\bar{w} + Ar\frac{\partial \bar{w}}{\partial r}\right) \\ = K\bar{w}\left(\sin\alpha\frac{\partial \bar{w}}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial \bar{w}}{\partial \alpha}\right) + A\frac{\partial \bar{w}}{\partial \alpha}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + 2A\bar{w} + Ar\frac{\partial \bar{w}}{\partial r}\right) \\ - A\frac{\partial \bar{w}}{\partial r}\left(\frac{\partial^2\varphi}{\partial r\,\partial \alpha} + Ar\frac{\partial \bar{w}}{\partial \alpha}\right),\end{aligned} \tag{A 2}$$

where $\nabla^4 = \nabla^2\nabla^2$.

We now recall the following identities:

$$\nabla^2(r\bar{w}) = r\nabla^2\bar{w}+2\frac{\partial\bar{w}}{\partial r}+\frac{\bar{w}}{r}, \tag{A 3}$$

$$\frac{\partial}{\partial r}(\nabla^2\bar{w}) = \nabla^2\left(\frac{\partial\bar{w}}{\partial r}\right)-\frac{1}{r^2}\frac{\partial\bar{w}}{\partial r}-\frac{2}{r^3}\frac{\partial^2\bar{w}}{\partial\alpha^2}, \tag{A 4}$$

so that we obtain

$$\nabla^2\left(r\frac{\partial\bar{w}}{\partial r}\right)-\frac{\partial}{\partial r}(r\nabla^2\bar{w}) = \nabla^2\bar{w}. \tag{A 5}$$

By applying these identities and by the use of (32) and its derivate,

$$\frac{\partial}{\partial r}(r\nabla^2\bar{w})+4 = \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\frac{\partial\bar{w}}{\partial\alpha}-\frac{\partial\varphi}{\partial\alpha}\frac{\partial\bar{w}}{\partial r}\right), \tag{A 6}$$

we finally obtain from (A 2) the simple expression (33).

Appendix B

The resulting equations of Murata *et al.* (1981) are the following (their equations (18)–(20)):

$$\nabla^2 f = -\omega, \tag{B 1}$$

$$\nabla^2\omega+\frac{1}{r}\frac{\partial(f,\omega)}{\partial(r,\varphi)} = 2D_1D_2-w\left(\frac{\partial w}{\partial r}\sin\varphi+\frac{1}{r}\frac{\partial w}{\partial\varphi}\cos\varphi\right), \tag{B 2}$$

$$\nabla^2 w+\frac{1}{r}\frac{\partial(f,w)}{\partial(r,\varphi)} = -D_1, \tag{B 3}$$

where D_1 is equivalent to $4K^{\frac{1}{2}}$, w corresponds to $\bar{w}K^{\frac{1}{2}}$, $\varphi = \alpha+\pi$ and D_2 is equal to $-2\beta^{\frac{1}{2}}$ (the minus sign is because Kao considers a right-handed helix).

The function f that appears in these equations is the true stream function in the non-orthogonal system adopted by Murata *et al.*, so that their non-orthogonal velocity components are

$$u = \frac{1}{r}\frac{\partial f}{\partial\varphi}, \quad v = -\frac{\partial f}{\partial r}, \tag{B 4}$$

and v does not correspond to the Kao orthogonal components v.

The transformations to the orthogonal components are (Murata *et al.*, equation (23))

$$U = u = \frac{1}{r}\left(\frac{\partial f}{\partial\varphi}\right), \quad V = v+\tfrac{1}{2}D_2 rw = -\frac{\partial f}{\partial r}+\tfrac{1}{2}D_2 rw, \quad W = w, \tag{B 5}$$

and now U and V are the u and v of Kao. It is interesting to note that the same equations are interpreted in the orthogonal system of coordinates as a derivation of the velocities from a modified stream function φ, and in the non-orthogonal system as a transformation of components. Anyway, the method used does not matter, and it is interesting to note that similar equations are obtained from following different routes.

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